Ordering of Random Walks: The Leader and the Laggard

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(Dated: February 1, 2008)

We investigate two complementary problems related to maintaining the relative positions of N random walks on the line: (i) the leader problem, that is, the probability $\mathcal{L}_N(t)$ that the leftmost particle remains the leftmost as a function of time and (ii) the laggard problem, the probability $\mathcal{R}_N(t)$ that the rightmost particle never becomes the leftmost. We map these ordering problems onto an equivalent $(N-1)$–dimensional electrostatic problem. From this construction we obtain a very accurate estimate for $\mathcal{L}_N(t)$ for $N=4$, the first case that is not exactly soluble: $\mathcal{L}_4(t) \propto t^{-\beta_4}$, with $\beta_4 = 0.91342(8)$. The probability of being the laggard also decays algebraically, $\mathcal{R}_N(t) \propto t^{-\gamma_N}$; we derive $\gamma_2 = 1/2, \gamma_3 = 3/8$, and argue that $\gamma_N \to N^{-1} \ln N$ as $N \to \infty$.

PACS numbers: 02.50.Cw, 05.40.-a, 05.50.+q, 87.18.Sn

I. INTRODUCTION

Consider N identical and independent random walkers that are initially located at $x_1(0) < x_2(0) < \cdots < x_N(0)$ on an infinite line [1]. There are many interesting questions that can be posed about the order of the particles. For example, what is the probability that *all* walkers maintain their relative positions up to time t, that is, $x_1(\tau) < x_2(\tau) < \cdots < x_N(\tau)$ for all $0 \leq \tau \leq t$? By mapping this *vicious random walk* problem onto the diffusion of a single effective particle in N dimensions and then exploiting the image method for the diffusion equation, this ordering probability was found [2, 3] to decay asymptotically as $t^{-\alpha_N}$ with $\alpha_N = N(N-1)/4$. Many additional aspects of this problem have been investigated within the rubrics of vicious random walks [2, 4, 5, 6, 7, 8, 9] and "friendly" random walks [10, 11].

In this work, we study two related and complementary random walk ordering problems. In the "leader" problem, we ask for the probability $\mathcal{L}_N(t)$ that the initially leftmost particle in a group of N particles remains to the left of all the other particles up to time t [12, 13, 14, 15, 16, 17]. In the "laggard" problem, we are concerned with the probability $\mathcal{R}_N(t)$ that the initially rightmost particle from a group of N particles never attains the lead (becomes leftmost). These two probabilities $\mathcal{L}_N(t)$ and $\mathcal{R}_N(t)$ decay algebraically

$$\mathcal{L}_N(t) \propto t^{-\beta_N} , \qquad \mathcal{R}_N(t) \propto t^{-\gamma_N} , \qquad (1)$$

as $t \to \infty$ and our basic goal is to determine the exponents β_N and γ_N.

These ordering problems arise in a variety of contexts. Physical applications include wetting phenomena [2, 3] and three-dimensional Lorentzian gravity [18]. A more probabilistic application is the ballot problem [12], where one is interested in the probability that the vote for a single candidate remains ahead of all the other candidates throughout the counting; this is just a restatement of the leader problem. Another example is that of the lamb and the lions [15, 16], in which one is interested in the survival of a diffusing lamb in the presence of many diffusing lions. In one dimension, a lamb that was initially in the lead must remain the leader to survive.

For the leader problem, exact results are known for small N only: $\beta_2 = 1/2$ and $\beta_3 = 3/4$ [4, 12, 19, 20], while $\beta_N \to \ln(4N)/4$ for large N [14, 15, 16, 17]. This slow increase arises because adding another particle has little effect on the survival of the leader when N is large. For $N \geq 4$, no exact results are available and one focus of our work is to obtain an accurate estimate of β_N for the case $N = 4$. We accomplish this by mapping the reaction onto an equivalent electrostatic potential problem due to a point charge within an appropriately-defined three-dimensional domain. This mapping provides both an appealing way to visualize the reaction process and an accurate estimate of the survival exponent β_4.

For the laggard problem, we employ the same method as in the leader problem to obtain $\gamma_3 = 3/8$. We also estimate the asymptotic behavior of $\mathcal{R}_N$ and find $\gamma_N \to N^{-1} \ln N$ as $N \to \infty$. As is expected, a laggard in a large population likely remains a laggard. Therefore the probability of remaining a laggard decays very slowly with time for large N.

In the next section, we review known results about the leader problem in a 3-particle system. In Sec. III, we outline the electrostatic formulation of the leader problem and then apply it to the 4-particle system in Sec. IV. A numerical solution of the pertinent Laplace equation gives $\beta_4 = 0.91342(8)$, a significant improvement over the previously-quoted estimate $\beta_4 \approx 0.91$ [13]. In Sec. V, we turn to the laggard problem and give an asymptotic estimate for the exponent γ_N. Concluding remarks and some open questions are given in Sec. VI.

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II. CONVENTIONAL APPROACH TO THE LEADER PROBLEM

We begin by mapping the original problem of N diffusing particles $x_1, x_2, \ldots, x_N$ on the line onto a single effective diffusing particle located at $\big(x_1, x_2, \ldots, x_N\big)$ in N-dimensional space. The particle order constraints on the line translate to bounding hyperplanes within which the effective particle is confined [2, 20]. The effective particle is absorbed if it hits one of these boundaries. For the leader problem, the explicit shape of these bounding hyperplanes can be easily worked out for the cases $N = 2$ and $N = 3$; we will later extend this analysis to the 4-particle system.

For 2 particles, their separation undergoes simple diffusion and the process terminates when the separation equals zero. Thus the survival probability of the leader decays as $t^{-1/2}$. To fix notation and ideas for later sections, we now study the 3-particle system. For a leader at $x_1(t)$ and particles at $x_2(t)$ and $x_3(t)$, we view these coordinates as equivalent to the isotropic diffusion of a single effective particle at $\big(x_1(t), x_2(t), x_3(t)\big)$ in three dimensions. Whenever this effective particle crosses the plane $A_{ij}:\ x_i = x_j$, the original walkers at x_i and x_j in one dimension have reversed their order. There are three such planes A_{12}, A_{13}, A_{23} that divide space into 6 domains, corresponding to the 3! possible orderings of the three walkers (Fig. 1(a)). These planes all intersect along the $(1, 1, 1)$ axis.

We may simplify this description by projecting onto the plane $x_1 + x_2 + x_3 = 0$ that contains the origin and is perpendicular to the $(1, 1, 1)$ axis. Now the plane A_{12} may be written parametrically as (a, a, b) and its intersection with the plane $x_1+x_2+x_3 = 0$ is the line $(a, a, -2a)$. Likewise, the intersections of A_{13} and A_{23} with the plane $x_1 + x_2 + x_3 = 0$ are $(a, -2a, a)$ and $(-2a, a, a)$, respectively (Figs. 1(b)).

The survival of the leader corresponds to the effective particle remaining within the adjacent domains 123 and 132 in Fig. 1(b). The background particles at x_2 and x_3 are allowed to cross, but the leader at x_1 always remains to the left of both x_2 and x_3. The union of these two domains defines a wedge of opening angle 120°. Since the survival probability of a diffusing particle within a wedge of arbitrary opening angle φ and absorbing boundaries decays asymptotically as $t^{-\pi/2\varphi}$ [21], we deduce the known result that the leader survival probability exponent is $\beta_3 = 3/4$.

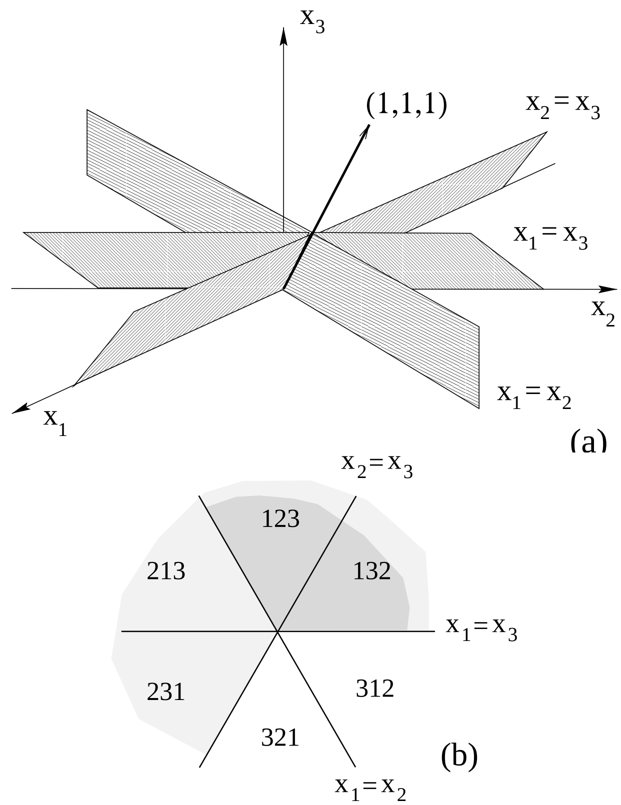

FIG. 1: The order domains for 3 particles in (a) the full 3-dimensional space, and (b) projected onto the subspace perpendicular to the $(1, 1, 1)$ axis. The notation ijk is shorthand for $x_i < x_j < x_k$. The allowed region corresponding to survival of a leader ($x_1 < x_2, x_3$) is indicated by the darker shading, while the lighter shaded region corresponds to the laggard problem ($x_3 \not< x_1, x_2$).

III. ELECTROSTATIC FORMULATION

For more than 3 particles, the domain for the effective particle is geometrically more complex and the corresponding solution to the diffusion equation does not seem tractable. We therefore recast the survival probability of the effective diffusing particle in terms of the simpler problem of the electrostatic potential of a point charge within the same geometric domain [20]. Let $S(t)$ be the survival probability of a diffusing particle within an infinite wedge-shaped d-dimensional domain with absorbing boundary conditions. Let $V(r)$ be the electrostatic potential due to a point charge within the same domain, with $V = 0$ on the boundary. Generically, these two quantities have the asymptotic behaviors

$$\begin{aligned} S(t) &\propto t^{-\beta}\,, \qquad t \to \infty\,, \\ V(r) &\propto r^{-\mu}\,, \qquad r \to \infty\,. \end{aligned} \tag{2}$$

More relevant for our purposes, these two quantities are simply related by [15, 16, 20]

$$\int_0^t S(t)\,dt \sim \int^{\sqrt{t}} V(r)\,r^{d-1}dr\,.$$

This equivalence arises because the integral of the diffusion equation over all time is just the Laplace equation. Thus the time integral of the survival probability has the same asymptotic behavior as the spatial integral of the electrostatic potential over the portion of the domain that is accessible by a diffusing particle up to time

t. Substituting the respective asymptotic behaviors from Eqs. (2), and noting that the allowed wedge domain for an N-particle system has dimension $N-1$, we obtain the fundamental exponent relation

$$\beta = \frac{\mu - N + 3}{2} \,. \tag{3}$$

Thus the large-distance behavior of the electrostatic potential within a specified domain with Dirichlet boundary conditions also gives the long-time survival probability of a diffusing particle within this domain subject to the same absorbing boundary conditions. From this survival probability, we can then determine the original ordering probability.

To illustrate this approach, let us determine the various ordering probabilities of 3 particles on the line in terms of the equivalent electrostatic problem. In fact, it is simpler to work backwards and find the equivalent ordering problem that corresponds to a specific wedge domain, For example, consider the $60°$ wedge 123 in Fig. 1. If the effective particle remains within this wedge, the initial particle ordering on the line is preserved. This corresponds to the vicious random walk problem in which no particle crossings are allowed. To obtain the asymptotic behavior of the potential of a point charge interior to this wedge, let us assume that $V(\mathbf{r}) \sim r^{-\mu} f(\varphi)$ as $r \to \infty$. Substituting this ansatz into the 2-dimensional Laplace equation, we obtain the eigenvalue equation $f''(\varphi) = -\mu^2 f(\varphi)$, subject to $f(\varphi) = 0$ on the wedge boundaries. For the $60°$ wedge, the solution with the smallest eigenvalue is $f(\varphi) = \sin(3\varphi)$. Thus $\mu = 3$, leading to the known result $\beta = (\mu - N + 3)/2 = 3/2$, for 3 vicious walkers.

IV. 4-PARTICLE SYSTEM

The state of the system may be represented by an effective diffusing particle in 4 dimensions. By projecting onto the 3-dimensional subspace $x_1 + x_2 + x_3 + x_4 = 0$ that is orthogonal to the $(1,1,1,1)$ axis, the order domains of the original particles can be reduced to 3 spatial dimensions. In this 3-dimensional subspace, the boundary $A_{12}: \; x_1 = x_2$ becomes the plane (a, a, b, c), with $2a + b + c = 0$. Likewise, A_{13} may be written parametrically as (a, b, a, c), where again $2a + b + c = 0$. Thus the locus $L_{123} \equiv A_{12} \cap A_{13}: \; x_1 = x_2 = x_3$ is the line $(a, a, a, -3a)$. This body diagonal joins the nodes 4 and $\overline{4}$ in Fig. 2. Along this axis, the original particle coordinates on the line obey the constraint $x_1 = x_2 = x_3$, with $x_4 < x_3$ on the half-axis closer to node 4 and $x_4 > x_3$ on the half-axis closer to node $\overline{4}$. A similar description applies to the axes $L_{124} = (a, a, -3a, a)$ between 3 and $\overline{3}$, $L_{134} = (a, -3a, a, a)$ between 2 and $\overline{2}$, and $L_{234} = (-3a, a, a, a)$ between 1 and $\overline{1}$.

The locus where $x_1 = x_2$ and $x_3 = x_4$ simultaneously, is the line $L_{12,34} \equiv A_{12} \cap A_{34} = (a, a, -a, -a)$. Likewise, $L_{13,24} = (a, -a, a, -a)$, and $L_{14,23} = (a, -a, -a, a)$. Viewed in the orthogonal 3-subspace, the 6 planes A_{ij}

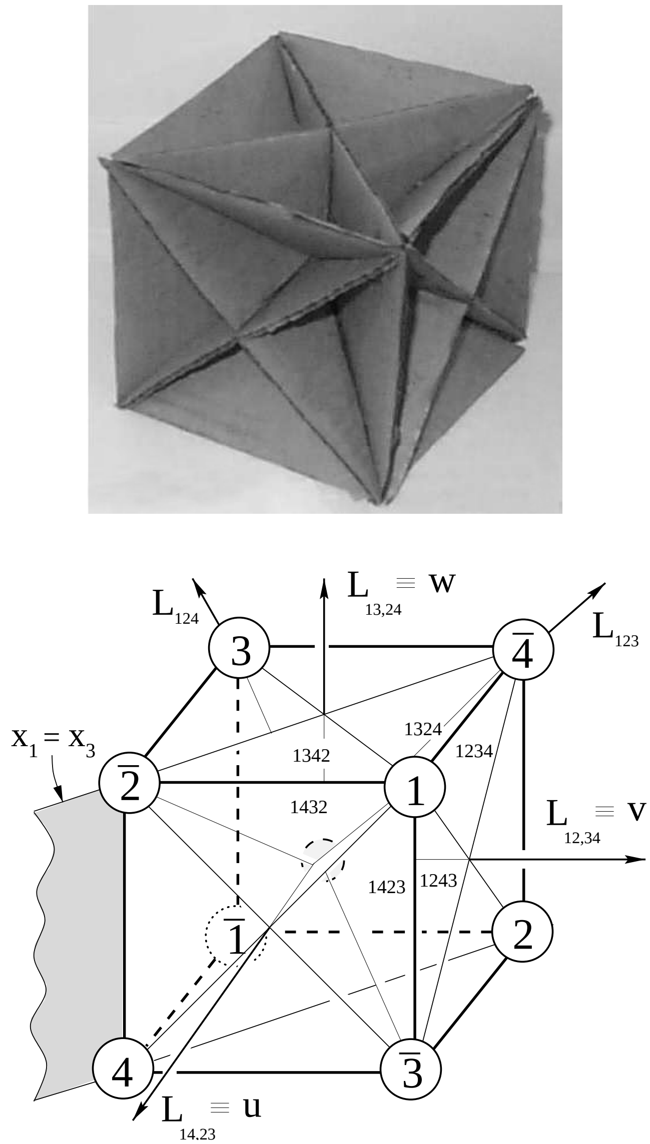

FIG. 2: Top: Cardboard model of the ordering domains for 4 particles on the line after projection into the subspace perpendicular to $(1,1,1,1)$. This structure consists of only 6 intersecting planes. Each plane bisects the cube and is defined by the equality of two coordinates. Lower: Schematic of the same system. The wedge $ijkl$ denotes the region where $x_i < x_j < x_k < x_l$. The union of six such ordering wedges, corresponding to the leader problem $x_1 < x_2, x_3, x_4$, are labeled. This domain is bounded by the rays $0\overline{2}$, $0\overline{3}$, $0\overline{4}$, where the origin is the dashed circle at the intersection of the axes $L_{14,23} \equiv u$, $L_{12,34} \equiv v$, and $L_{13,24} \equiv w$. One constraint plane, $x_1 = x_3$, is shown shaded (outside the cube).

intersect in 7 lines (4 L_{ijk} and 3 $L_{ij,kl}$), and divide the subspace into 24 semi-infinite wedges, as shown in Fig. cube(b). Each of these wedges corresponds to one of the 4! orderings of the walkers in one dimension.

We first illustrate the electrostatic formulation of this system for the vicious random walk problem. Since the initial particle order is preserved, the effective dif-

fusing particle remains within a single wedge $ijkl$ in Fig. 2. As outlined in the previous section, the survival probability of this effective particle corresponds to the electrostatic potential of a point charge within this one wedge, with the boundary surfaces held at zero potential. To solve this electrostatic problem, it is convenient to place the point charge at the symmetric location $(u, v, w) = (0, 1, 1/2)$ within the wedge 1234, where the (u, v, w) axes are defined in Fig. 2. From the image method, the potential due to a point charge within one wedge is equivalent to the potential of an array of 24 symmetrically-placed point charges consisting of the initial charge and 23 image charges, with positive images at $-(0, 1, 1/2)$, $\pm(0, 1, -1/2)$, $\pm(\pm 1/2, 0, 1)$, $\pm(1, \pm 1/2, 0)$, and negative images at $\pm(\pm 1/2, 1, 0)$, $\pm(0, \pm 1/2, 1)$, and $\pm(0, 1, \pm 1/2)$. Using Mathematica the asymptotic behavior of the potential in wedge 1234 (where the original charge is placed) due to this charge array is

$$V(r) = a_1 r^{-7} + a_2 r^{-11} + a_3 r^{-13} + a_4 r^{-15} + \cdots .$$

(The coefficients a_i depend on the location of the charge and on the orientation of $\mathbf{r}$.) Using the exponent relation (3), the asymptotic survival probability of 4 vicious walkers is given by

$$S(t) = b_1 t^{-3} + b_2 t^{-5} + b_3 t^{-6} + b_4 t^{-7} + \cdots .$$

The leading behavior confirms the known result [2, 3], and we obtain the form of the corrections as well. As an amusing aside, notice that an r^{-7} dependence for the potential is normally achieved by a 64-pole charge configuration. The high symmetry of the 24 charges in the ordering problem leads to a multipole field normally associated with at least 64 point charges.

We next turn to the leader problem. This system corresponds to the electrostatic problem within the combined domain of the 6 wedges marked 1234, 1243, 1423, 1432, 1342, and 1324, in Fig. 2. The resulting domain is a tetrahedral corner, with its apex at the center of the cube, that is flanked by the rays $0\overline{2}$, $0\overline{3}$, and $0\overline{4}$. Despite the simplicity of this domain, we are unable to solve this electrostatic problem analytically and we have instead studied the problem numerically.

In the allowed region of the cube in Fig. 2, we discretize space and solve the electrostatic potential of a point charge by using successive over-relaxation, with the domain boundaries at zero potential. For simplicity, the charge is chosen to be at the symmetric point $(1, 1, 1)$. For the outer faces of the cube we use two different boundary conditions: (a) absorbing ($V = 0$), and (b) reflecting, ($dV/dn = 0$, where n is the direction normal to the surface). The true potential – that of the infinite wedge – lies between these two extremes. We also exploit the symmetry about the $(1, 1, 1)$ diagonal and use only the domain marked 1234 in Fig. 2, with absorbing boundary conditions on the plane $34\overline{3}\overline{4}$, and reflecting boundary conditions on the planes $12\overline{1}\overline{2}$ and $14\overline{1}\overline{4}$. As already discussed, the outer cube face, $1\overline{3}2\overline{4}$, is taken to be reflecting or absorbing. This space savings allows us to carry out computations for a cube of 500 lattice spacings on a side.

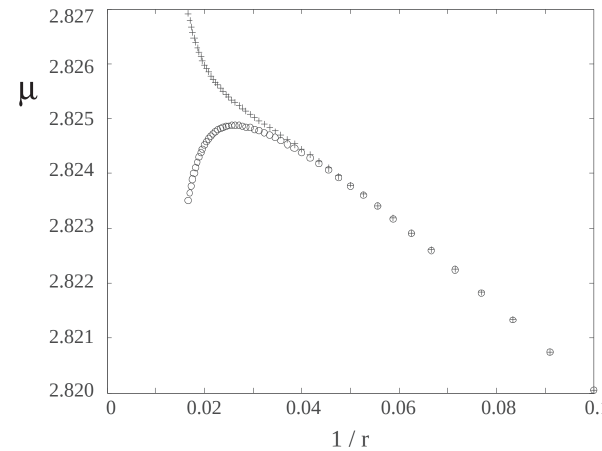

FIG. 3: Local exponent $\mu(r)$, as a function of $1/r$ for the tetrahedral wedge with absorbing $(+)$ and reflecting $(\circ)$ finite-size boundary conditions at the outer face of the cube.

While the power-law decay of the potential sets in quickly, the finite-size effect is pronounced and it is perceptible already at 25 lattice spacings away from the charge. This is the primary limitation on the accuracy of our exponent estimate. Nevertheless, the local exponent $\mu(r) = -d \ln V(r)/d \ln r$ varies only at the fourth digit (Fig. 3). The approach of the local exponent to the asymptotic limit also suggests that $V(r)$ has the form $V(r) \sim r^{-\mu} + A r^{-4}$. Assuming that this is the case, extrapolation of the data in Fig. 3 gives $\mu = 2.82684 \pm 0.00016$, where the error bar is the difference in the extrapolated value of $\mu(r)$ from the two different boundary conditions. From the exponent correspondence given in Eq. (3), we thereby obtain, for the lead probability,

$$\mathcal{L}_4(t) \sim t^{-\beta_4} + A t^{-3/2} , \qquad \beta_4 = 0.91342(8) , \tag{4}$$

It is hard to match this numerical accuracy with that from direct simulations of the survival of the leader. We simulated 10^9 realizations of the system; this gives extremely linear data for the time dependence of the leader survival probability on a double logarithmic scale. To estimate the exponent β_4, we computed the local slopes of the survival probability versus time in contiguous time ranges between t and $1.5t$ when plotted on a double logarithmic scale. These local exponents are plotted against $1/\ln_{1.5} t$ (Fig. 4). The results are compatible but much less accurate than Eq. (4).

V. THE LAGGARD PROBLEM

In the laggard problem, we study the probability that the initially rightmost particle at x_N has never been the

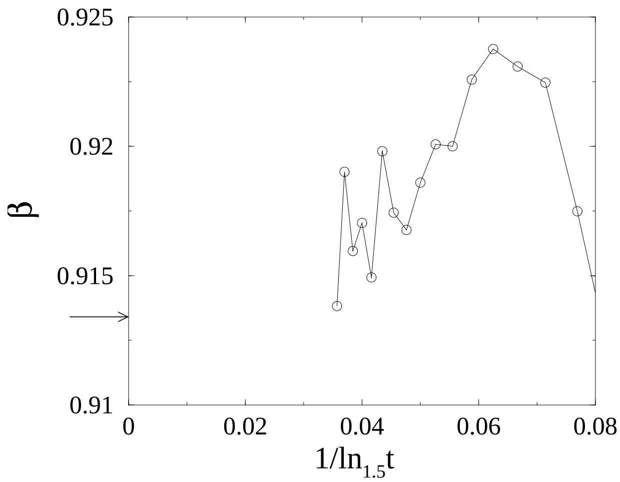

FIG. 4: Direct simulation results for the local exponent in the survival probability for 10^9 configurations. The arrow indicates our estimate for β_4 from Eq. (4).

leader during the time interval $(0,t)$. The laggard problem can also be recast into the diffusion of a single effective particle within an N–dimensional wedge-shaped region, with absorbing domain boundaries. This mapping leads to the basic conclusion that every particle that is initially not in the lead exhibits the same asymptotic behavior as the last particle. Indeed, for any particle i initially at x_i, the regions $x_i \not< x_1,\ldots,x_{i-1},x_{i+1},\ldots,x_N$ are isomorphic. The initial condition merely fixes the location of the effective particle in this allowed region. Another fundamental observation is that the allowed regions of the effective particle for the leader and the laggard problems are *complementary* for all N.

For two particles, the probability that the laggard does not become the leader obviously decays as $t^{-1/2}$, *i.e.*, $\gamma_2 = 1/2$. The case $N=3$ is more interesting but also solvable. The condition that a particle initially at x_3 never attaining the lead $(x_3 \not< x_1, x_2)$ is equivalent to the effective particle remaining with the lighter shaded region in Fig. 1(b). Since the opening angle of the resultant wedge is $\varphi = 4\pi/3$, the corresponding survival probability decays as $t^{-3/8}$, implying that $\gamma_3 = 3/8$.

We have performed direct numerical simulations of the process to estimate the exponent γ_N for $N = 3,4,5,$ and 6. Each simulation is based on 10^6 realizations in which $N-1$ particles are initially at the origin, while the laggard is at $x=1$. Each simulation is run until the laggard achieves the lead or 10^5 time steps, whichever comes first. From the survival probability, we estimate $\gamma_N \approx 0.35, 0.30, 0.26,$ and 0.23 for $N = 3-6$, respectively. Since we know that $\gamma_3 = 0.375$, the discrepancy of 0.025 between the simulation result and the theory is indicative of the magnitude of systematic errors in this straightforward numerical approach.

While it appears difficult to determine the exponent γ_N analytically for general $N > 3$, the situation simplifies in the large N limit because the position of the leader becomes progressively more deterministic. Indeed, the probability density $P_N(y,t)$ that the leader is located at distance y from the origin (assuming that all particles are initially at the origin) is [16]

$$P_N(y,t) = \frac{N\, e^{-y^2/4Dt}}{\sqrt{4\pi Dt}} \left[1 - \frac{1}{2}\,\mathrm{erfc}\left(-\frac{y}{\sqrt{4Dt}}\right)\right]^{N-1},$$

where $\mathrm{erfc}(x) = 1 - \mathrm{erf}(x)$ is the complementary error function. Performing a large N analysis, we find that the probability density $P_N(y,t)$ approaches a Gaussian

$$P_N(y,t) \to \frac{1}{\sqrt{2\pi\sigma^2}}\, \exp\left\{-\frac{(y+y_*)^2}{2\sigma^2}\right\}, \tag{5}$$

with the mean and the variance of this distribution given by

$$y_* = z\sqrt{4Dt}\,, \qquad \sigma^2 = \frac{Dt}{z^2}\,, \tag{6}$$

where z is determined from the transcendental relation

$$z\, e^{z^2} = \frac{N}{\sqrt{4\pi}}. \tag{7}$$

Consequently, the parameter z diverges as $z \approx \sqrt{\ln N}$ when $N \to \infty$. The ratio of the dispersion to the mean displacement thus decreases as $\sigma/y_* \approx (2\ln N)^{-1}$, so that the position of the leader indeed becomes more deterministic as $N \to \infty$.

Therefore in the large N limit we can assume that the leader is moving deterministically and its position is given by $-y_*(t)$. Then the probability that the laggard never achieves the lead is equivalent to the probability that a diffusing particle initially at the origin does not overtake a receding particle whose position is varying as $-y_*(t)$. This corresponds to the solution to the diffusion equation in the expanding region $x \in (-y_*(t), \infty)$ with an absorbing boundary condition at the receding boundary $x = -y_*$. When $y_* \propto t^{1/2}$, this diffusion equation can be solved exactly by reducing it to a parabolic cylinder equation in a fixed region. However, in the limit $t \to \infty$, we can obtain asymptotically correct results much more simply. Because the absorbing boundary recedes from the laggard particle relatively quickly, we solve the problem by assuming that the density $P(x,t)$ of the laggard particle approaches a Gaussian with yet unknown amplitude $\mathcal{R}_N(t)$ [15, 16]:

$$P(x,t) = \frac{\mathcal{R}_N(t)}{\sqrt{4\pi Dt}}\, \exp\left\{-\frac{x^2}{4Dt}\right\}. \tag{8}$$

Although this distribution does not satisfy the absorbing boundary condition, the inconsistency is negligible since the exponential term in Eq. (8) is of order N^{-1} at the boundary $y = y_*$.

The probability $\mathcal{R}_N(t)$ is now found self-consistently by equating the "mass" loss to the flux:

$$\frac{d\mathcal{R}_N}{dt} = D\frac{\partial P}{\partial x}\Big|_{x=-y_*} \tag{9}$$

Using Eq. (8) to compute the flux we convert Eq. (9) into

$$\frac{d\mathcal{R}_N}{dt} = -\frac{z^2}{N}\,\frac{\mathcal{R}_N}{t}, \tag{10}$$

from which the exponent γ_N is z^2/N. This is of course valid only in the large N limit. Taking this limit in Eq. (7) we obtain

$$\gamma_N = \frac{\ln N}{N} - \frac{\ln\ln N}{2N} + \ldots. \tag{11}$$

Thus as N gets large, the probability that the laggard never attains the lead decays extremely slowly with time. This fits with the naive intuition that if the number of particles is large a laggard initially is very likely to remain a laggard. Each additional particle makes it even less likely that the laggard could achieve the lead. Amusingly, this asymptotic exponent predictions is numerically close to the previously-quoted results from direct numerical simulations of the laggard problem.

VI. CONCLUDING REMARKS

We investigated two dual random walk ordering problems in one dimension: (i) what is the probability that a particle, that is initially in the lead, remains in the lead and (ii) what is the probability that a particle, that is initially not in the lead, never achieves the lead? These problems are most interesting in one spatial dimension because of the effective correlations between the interacting particles. These correlations are absent in two dimensions and greater, so that an N-particle system reduces to a 2-particle system [9, 15, 16].

We determined the respective exponents β_N and γ_N associated with the lead and laggard probabilities for general N. Both exponents can be determined by elementary geometric methods for $N = 2$ and 3 and by asymptotic arguments for $N \to \infty$. Our new results are the following: (i) a precise estimate for β_4 and (ii) the large-N behavior of γ_N.

A simple generalization is to allow each particle i to have a distinct diffusion coefficient D_i. The exponents β_N and γ_N will now depend on the diffusion coefficients, except for $N = 2$, where β_2 and γ_2 always equal 1/2. The case $N = 3$ is still solvable by introducing rescaled coordinates $y_i = x_i/\sqrt{D_i}$ to render the diffusion of the effective particle isotropic, after which the mapping to the wedge can be performed straightforwardly. We thus find

$$\beta_3 = \left\{2 - \frac{2}{\pi}\cos^{-1}\frac{D_1}{\sqrt{(D_1+D_2)(D_1+D_3)}}\right\}^{-1},$$
$$\gamma_3 = \left\{2 + \frac{2}{\pi}\cos^{-1}\frac{D_3}{\sqrt{(D_1+D_3)(D_2+D_3)}}\right\}^{-1}.$$

An amusing special case is the case of a stationary laggard, for which $\gamma_3 = 1/3$.

Finally, it is also worth mentioning a promising development to solve the diffusion equation in the domains defined by the ordering of one-dimensional random walks. This is the recent discovery of deep connections between vicious walkers and random matrix theory [6, 7, 8]. These allow one to not only re-derive the exponent α_N of the original vicious random walk problem, but also lead to many new results. It would be extremely useful if these techniques could be extended to the leader and laggard problems.

Acknowledgments

We are grateful to NSF grants PHY-0140094 (DbA) and DMR9978902 (PLK and SR) for partial financial support of this research.

[1] We assume that all random walkers have the same diffusion coefficient D if not stated otherwise. For distinct diffusivities D_i for each particle, one may rescale each coordinate by $x_i \to y_i = x_i/\sqrt{D_i}$ so that the motion of the effective particle is again isotropic.

[2] M. E. Fisher, "Walks, walls, wetting, and melting", J. Stat. Phys. **34**, 667–729 (1984).
[3] D. A. Huse and M. E. Fisher, "Commensurate wetting, domain walls, and dislocations", Phys. Rev. B **29**, 239–270 (1984).
[4] M. E. Fisher and M. P. Gelfand, "The reunions of three dissimilar vicious walkers", J. Stat. Phys. **53**, 175 (1988).
[5] P. J. Forrester, "Exact solution of the lock step model of vicious walkers", J. Phys. A **23**, 1259–1273 (1990).
[6] D. J. Grabiner, "Brownian motion in a Weyl chamber, non-colliding particles, and random matrices", Ann. Inst. H. Poincaré. Prob. Stat. **35**, 177–204 (1999).
[7] J. Baik, "Random vicious walkers and random matrices", Commun. Pure Appl. Math. **53**, 1385–1410 (2000).
[8] M. Katori and H. Tanemura, "Scaling limit of vicious walks and two-matrix model", Phys. Rev. E **66**, 011105 (2002).
[9] J. Cardy and M. Katori, "Families of vicious walkers", *cond-mat*/0208228 (2002).
[10] A. J. Guttmann and M. Vöge, "Lattice paths: vicious walkers and friendly walkers", J. Statist. Plann. Inf. **101**, 107–131 (2002).
[11] G. Krattenthaler, A. J. Guttmann, and X. G. Viennot, "Vicious walkers, friendly walkers and Young tableaux: II With a wall", J. Phys. A **33**, 8835–8866 (2000).
[12] H. Niederhausen, "The ballot problem with three candidates", Eur. J. Combinatorics **4**, 161–167 (1983).
[13] M. Bramson and D. Griffeath, "Capture problems for coupled random walks," in *Random Walks, Brownian*

Motion, and Interacting Particle Systems: A Festschrift in Honor of Frank Spitzer, R. Durrett and H. Kesten, eds., pp. 153–188 (Birkhäuser, Boston, 1991).

[14] H. Kesten, "An absorption problem for several Brownian motions", in *Seminar on Stochastic Processes, 1991*, E. Çinlar, K. L. Chung, and M. J. Sharpe, eds. (Birkhäuser, Boston, 1992).

[15] P. L. Krapivsky and S. Redner, "Kinetics of a diffusive capture process: Lamb besieged by a pride of lions", J. Phys. A **29**, 5347–5357 (1996).

[16] S. Redner and P. L. Krapivsky, "Capture of the lamb: Diffusing predators seeking a diffusing prey", Am. J. Phys. **67**, 1277–1283 (1999).

[17] Numerical simulations also suggest a logarithmic dependence on N, but with a different prefactor. See P. Grassberger, "Go with the winners: a general Monte Carlo strategy", Comput. Phys. Commun. **147**, 64–70 (2002).

[18] B. Dittrich and R. Loll, "A hexagon model for 3D Lorentzian quantum cosmology", *hep-th*/0204210 (2002).

[19] D. ben-Avraham, "Computer simulation methods for diffusion-controlled reactions", J. Chem. Phys. **88**, 941 (1988).

[20] S. Redner, *A Guide to First-Passage Processes* (Cambridge University Press, 2001).

[21] H. S. Carslaw and J. C. Jaeger, *Conduction of Heat in Solids* (Oxford University Press, Oxford, U. K., 1959).